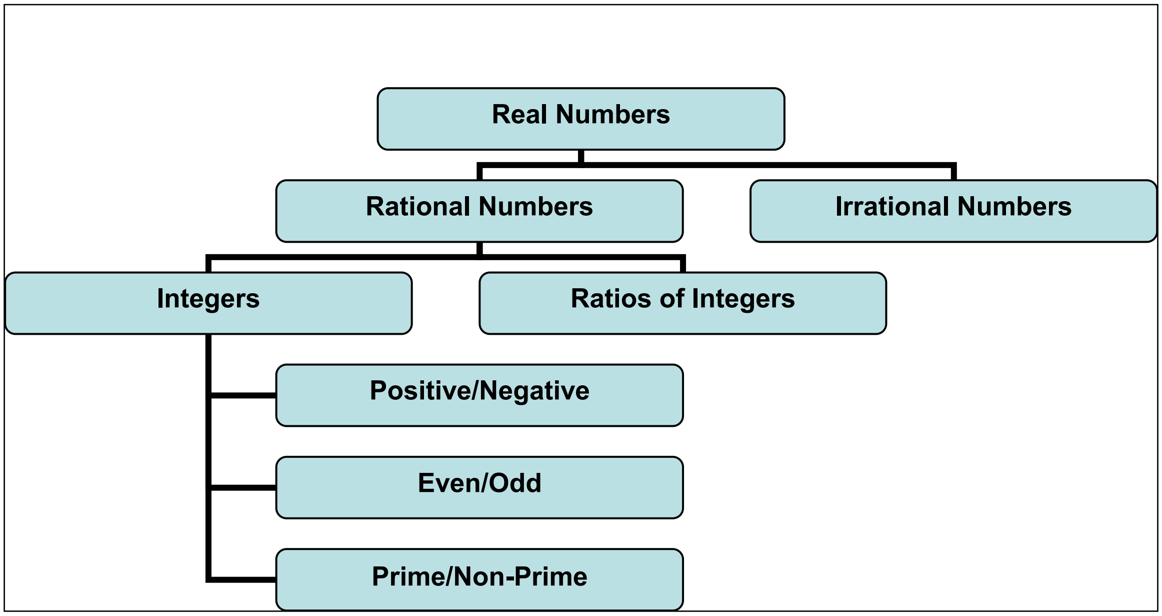
Numbers



Integer	Any number in the set {. . . -3, -2, -1, 0, 1, 2, 3, . . . } Remember that zero is an integer. Remember that big negative numbers are smaller.
Positive integer	Any of the numbers: 0, 1, 2, 3, …
Natural number	The numbers 1, 2, 3, …
Negative integer	Any of the numbers: …-3, -2, -1
Whole numbers	A positive integer
Rational number	A number which can be expressed as the ratio of two integers. Always results in a repeating decimal.
Irrational number	A number which cannot be expressed as the ratio of two integers. A number which has a non-repeating decimal
	Examples: $\sqrt{2} = 1.4142\ldots$, $e = 2.718281828$, $\pi = 3.14159$
Fraction	The ratio of two numbers, portions of whole numbers
Simple fraction	A ratio where the numerator and denominator are integers
Decimal number	A number written with a decimal point
Decimal fraction	The digits in a number written to the right of the decimal point
Real number	The set of rational and irrational numbers

Fractions

It is a good idea to memorize the following fractions because so many other fractions are formed from them.

1/2	1/4	1/8	1/16	1/32	1/64
.5	.25	.125	.0625	.03125	.015625

1/3	2/3	1/5	2/5	3/5	4/5
.3333…	.6666…	.2	.4	.6	.8

1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9
.500	.333	.250	.200	.167	.143	.125	.111

1/.2	10/3	10/4	10/5	10/6	10/7	10/8	10/9
5.00	3.33	2.50	2.00	1.67	1.43	1.25	1.11

$$\frac{23}{5} = \frac{23\times 2}{10} \qquad \frac{832}{25} = \frac{832\times 4}{10} \qquad \frac{856}{125} = \frac{856\times 8}{1000} \qquad \frac{120}{1.67} = \frac{120\times 6}{10}$$

Common Subtractions: note the pattern 1/(product of denominators)

1/2 – 1/3 = 1/6
1/3 – 1/4 = 1/12
1/4 – 1/5 = 1/20
1/5 – 1/6 = 1/30

Ordering of Fractions

First, try ranking them relative to something you know: 1/4, 1/2, and 3/4
Second, convert to lcd and compare numerators

$$\cdots < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{3}{7} < \frac{4}{9} < \frac{1}{2} < \frac{5}{9} < \frac{4}{7} < \frac{3}{5} < \frac{5}{8} < \frac{2}{3} < \frac{5}{7} < \frac{3}{4} < \frac{7}{9} < \frac{4}{5} < \frac{5}{6} < \frac{6}{7} < \frac{7}{8} < \frac{8}{9}$$

For example, what is the middle value of the fractions: 17/24, 1/2, 3/8, 3/4, 9/16. If you know your fractions really well you can just rank them, but it might help to rank them relative to 1/2 since it is easy to determine on which side of 1/2 they fall. Finally, convert them to their lcd (34/48, 24/48, 18/48, 36/48, 27/48) and then rank the numerators, which are just integers. Therefore the middle one is 27/48 or 9/16. This is the safest but most time consuming.

Irrational Numbers – Non-repeating Decimals

$\sqrt{1}$	1.0000
$\sqrt{2}$	1.4142
$\sqrt{3}$	1.7321
$\sqrt{4}$	2.0000
$\sqrt{5}$	2.2361
$\sqrt{6}$	2.4495
$\sqrt{7}$	2.6458
e	2.7183
$\sqrt{8}$	2.8284
$\sqrt{9}$	3.0000
π	3.1416
$\sqrt{10}$	3.1623

Multiplication Tables Beyond 10

	11	12	13	14	15	16	17	18	19	20
11	121	132	143	154	165	176	187	198	209	220
12	132	144	156	168	180	192	204	216	228	240
13	143	156	169	182	195	208	221	234	247	260
14	154	168	182	196	210	224	238	252	266	280
15	165	180	195	210	225	240	255	270	285	300
16	176	192	208	224	240	256	272	288	304	320
17	187	204	221	238	255	272	289	306	323	340
18	198	216	234	252	270	288	306	324	342	360
19	209	228	247	266	285	304	323	342	361	380
20	220	240	260	280	300	320	340	360	380	400

Powers of 2 and Common Squares

2^0	1	0	0		
2^1	2	1	1	11	121
2^2	4	2	4	12	144
2^3	8	3	9	13	169
2^4	16	4	16	14	196
2^5	32	5	25	15	225
2^6	64	6	36	16	256
2^7	128	7	49	17	289
2^8	256	8	64	18	324
2^9	512	9	81	19	261
2^{10}	1024	10	100	20	400

Notice that the difference between two squares is always an odd number 2n+1.

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

This occurs because the square of an odd is odd and the square of an even is even.

Fast Addition

First add tens then add ones

38	49	89
51	17	37
80 + 9	50 + 16	110 + 16

Look for patterns of ten and add them first:

$2 + 9 + 2 + 5 + 1 + 3 + 2 + 8 + 8 = (9+1)+(5+2+3)+(8+2)+(8+2) = 4 \times 10 = 40$

37	$10+9 = 19$
24	$(10+12)\times 10 = 220$
82	$= 239$
96	
239	

Fast Percentages

8040 is approximately what percent of the total of the following numbers?

92
26048
53121
420
1401

Just add 1000's to get 80, and then take 8/80 to get approximately 10%

Another view of multiplication

$18\times 660 = 6600 + 5280 = 11880$ $42\times 38 = 1260 + 336 = 1596$

Possible shortcuts with multiples of 5%

5%	5/100	30%	3/10	55%	11/20	80%	4/5
10%	1/10	35%	7/20	60%	3/5	85%	17/20
15%	3/20	40%	2/5	65%	13/20	90%	9/10
20%	1/5	45%	9/20	70%	7/10	95%	19/20
25%	1/4	50%	1/2	75%	3/4	100%	1

Shortcuts with 9x%

$1/.9x \approx 1.0y$, where the digit $y = 10-x$. This is a good approximation for most situations.
Note that 90% of x can be written as $x - .1x$.

Number Theory

Properties of Integers	If x and y are integers, $x \neq 0$, then x is **divisor** of y if $\frac{y}{x} = n$ or $y = xn$ where n is some integer other than 1. y is **divisible** by x y is a **multiple** of x x is a **factor** of y
Numerator	The top part of a fraction. In the fraction y/x, y is the numerator.
Denominator	The bottom part of a fraction, In the fraction y/x, x is the denominator.
Quotient q Remainder r	If x and y are integers, then a unique quotient and remainder exist such that y/x can be written as $y = qx + r,\ 0 \le r < x$ Notice that y is divisible by x iff r = 0. Notice that if a number divides y and x it must also divide r! Notice that $\frac{y-r}{qx} = 1$
Even integer	Any integer divisible by 2: 2n+2, 2n+4, … {… -2, 0, 2, 4, 6, …} Note that zero is even
Odd integer	Any integer not divisible by 2: 2n+1, 2n+3, 2n+5, … {… -3, -1, 1, 3, 5, …}
Even/Odd combos	E+E = E E×E = E, an even number of evens are added O+O = E O×O = O, an odd number of odds are added E+O = O E×O = E, an even number of odds are added E+E+E = E E×E×E = E E/E = E or O (100/4 or 100/10) O+O+O = O O×O×O = O E/O = E E+E+O = O E×E×O = E O/E = not an integer E+O+O = E E×O×O = E O/O = O When adding, you need an odd number of odds to make odd. When multiplying, you need all odds to make odd. When confused just substitute 1, 2, 3 and 2, 3, 4 and test the result. When you multiply a consecutive sequence of numbers, the result can never be odd. (1×2×3 = 6) (2×3×4 = 24) Note: if n is an integer n(n+1) = E since O×E = E×O = E Note: the three nbr sequence n, n+1, n+2 is always OEO or EOE
Prime number	An integer which is only divisible by 1 and itself. Even numbers are never prime numbers. Two is the most common prime number. Every integer is either a prime number or can be expressed as the product of prime numbers.
First 30 primes **KNOW THE FIRST 10**	2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97 101 103 107 109 113

Fundamental Theorem of Arithmetic	Any integer can be written as a product of prime numbers. Some of the prime numbers may be raised to a power.
Divisibility Rules	÷2 if units position is even ÷3 sum of digits ÷3 ÷4 last two digits ÷4 ÷5 last digit is 0 or 5 ÷6 divisible by 2 and 3 ÷7 no rule ÷8 last three digits ÷8 or 000 ÷9 sum of digits ÷9 ÷10 last digit is zero ÷11 sum of digits in even places equals the sum of digits in the odd places or when the difference between these two sums is a multiple of 11. ÷12 sum of digits is multiple of 3 and last two digits are a multiple of 4 ÷25 last two digits 00 or multiple of 25, compare to 4 ÷125 last three digits 000 or multiple of 125, compare to 8
lcm Lowest k such that $\frac{k}{x}, \frac{k}{y}, \frac{k}{z}$ are all integers	**Least (Lowest) Common Multiple** of two integers x and y, lcm (x,y), is the **smallest** integer which is divisible by x and y. The lcm of two integers x and y can be found by factoring each number into prime factors. The lcm is the product of the prime factors, each raised to the highest power in which it occurs. Consider 36 and 120. The prime factors of 36 are $2^2 \times 3^2$, and the prime factors of 120 are $2^3 \times 3 \times 5$. The prime factors are 2, 3, and 5, but we must include the **highest power** to which 2 and 3 are raised. Thus the lcm of 36 and 120 is $2^3 \times 3^2 \times 5 = 8 \times 9 \times 5 = 360$. Note that xy is always divisible by x and y, but x or y are seldom the lcm of (x,y).
lcd ≡ lcm	**Least (Lowest) Common Denominator** (lcd) is the lcm applied to the denominators of fractions. $$\frac{1}{x}+\frac{1}{y}=\frac{\frac{lcm(x,y)}{x}+\frac{lcm(x,y)}{y}}{lcm(x,y)}$$ Notice that if lcm(x,y) = xy, then the best we can do is $$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$$
gcd Method 1 Prime Factor Method Largest k such that	Greatest Common Divisor (gcd) of x, y, and z is the **largest** number (integer) which evenly divides into x, y, and z. The procedure for finding the gcd is similar to that of finding the ldc with a slight twist. With the lcm, we used the highest power of the prime. With gcd, we use the **lowest power** of the prime. (1) Find all the prime factors and there powers for each number. (2) For each set of prime factors, find the common factors from each set. (3) These common

$\frac{x}{k}, \frac{y}{k}, \frac{z}{k}$ are all integers	factors will be the gcd. The remaining factors (the uncommon factors) will form three new numbers x', y', and z' which when multiplied by the gcd will produce their original values. Thus, x'×gcd = x, y'×gcd = y, and z'×gcd = z. This is all very confusing, and it is best illustrated with an example. Consider the three numbers 36,120, and 48. The procedure works as follows: 36 120 48 $2^2 \times 3^2$ $2^3{\times}3{\times}5$ $2^4{\times}3$ Step 1 $\mathbf{2^2{\times}3}{\times}3$ $\mathbf{2^2{\times}3}{\times}2{\times}5$ $\mathbf{2^2{\times}3}{\times}2^2$ Step 2 3 10 4 Step 3: x', y', z' 12×3 12×10 12×4 GCD = 12 = 36/3 = 120/10 =48/4
gcd Method 2 Division Method Euclidean Algorithm	The Euclidean Algorithm makes use of the fact that n is a divisor of y and x iff n is a divisor of y and qx + r, which means that n must be a divisor of r as well. That is, gcd (y,x) = gcd (x,r). If we apply this process recursively until r = 0, the last denominator will be the gcd. Consider 36 and 120. 120/36 = 3 r 12, 36/12 = 3 r 0; therefore, gcd (36,129) = 12. If more than two numbers are involved this process can be applied in successive pairs until all pairs are matched. Note that if x and y are prime numbers, gcd (x,y) = 1. Similarly, if gcd(x,y) = 1, then x and y must be relatively prime; that is, have no common factors.
lcm from gcd	For large numbers, one can use the gcd to find the lcm. If gcd(x,y) =d, then two numbers a and b exist such that x = da and y = db. Therefore, xy = dadb. lcm(x,y) = lcm(da, db) = dab $lcm(x,y) = \frac{dadb}{dab} = \frac{xy}{\gcd(x,y)}$

Decimals

$n = d_1 + d_2 10 + d_3 100 + \ldots$ if n_1 = ab and n2 = cd, then $n_1 = a10 + b$ $n_2 = c10 + d$ $n_1 + n_2 = (a+c)10 + (b+d)$	If n is some decimal number with digits $d_1 d_2 d_3$.., then the number can be written as the sum of the digits multiplied by the power of ten corresponding to the position of the digit.
$d = \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \cdots$ d	If d is a decimal fraction then the number can be written as the sum of the digits divided by the corresponding power of ten.

Repeating Decimals

All repeating decimals can be expressed as the ratio of two integers in fraction form. Similarly, all integer fractions (rational numbers) can be expressed as repeating decimals. Consider the number 0.7272…The decimals 72 repeat infinitely, and the number is written as $.\overline{72}$; that is, the decimals that repeat are written with a bar over them. Several examples will be used to illustrate the process of converting repeating decimal numbers to fractions. The method requires that the number be multiplied by a power of ten which will include all of the repeating decimals and then subtracting a power of ten which will remove the repeating decimal.

Decimal	Method	Fraction
$.\overline{72}$	$100n - n = 72.\overline{72} - .\overline{72} = 72$	$\frac{72}{99}$
$.1\overline{72}$	$1000n - 10n = 172.\overline{72} - 1.\overline{72} = 172 - 1 = 171$	$\frac{171}{990}$
$.53\overline{653}$	$10000n - 100n = 53653.\overline{653} - 53.\overline{653} = 53653 - 53 = 53600$	$\frac{53600}{99900} = \frac{536}{999}$
$.0\overline{625}$	$10000n - 10n = 625.\overline{625} - .\overline{625} = 625$	$\frac{625}{9990}$

Number of 9's = number of recurring digits
Number of 0's = number of nonrecurring digits

Algorithm:

1. For the numerator, take all the digits up to and including the repeating decimals and from that number subtract the number formed by taking the non-recurring digits.
2. For the denominator, construct a number beginning with as many nines as there are recurring digits and ending with as many zeros as there are nonrecurring digits.

Notice the somewhat amazing result that $.3\overline{9} = \frac{39-3}{90} = \frac{36}{90} = \frac{2}{5}$.

Addition and Multiplication

a + b = b + a	Commutative Law of Addition
ab = ba	Commutative Law of Multiplication
a + (b+c) = (a+b) + c	Associative Law of Addition
a(bc) = (ab)c	Associative Law of Addition
a(b+c) = ab + ac	Distributive Law of multiplication over addition
a + 0 = a	Identity operation for addition
a × 1 = a	Identity operation for multiplication
a × 0 = 0	Multiplication with zero
a - a = 0	Addition/Subtraction with zero
a / a = 1	Multiplicative inverse

Fractions

$\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$	Cross products or proportions where a:b = c:d
$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	Common denominator
$\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$	Addition with different denominators
$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$	Multiplication is numerator times numerator and denominator times denominator.
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$	Division can be converted to multiplication by inverting the denominator fraction.
$\frac{na}{nb} = \frac{a}{b}$	Common factors cancel and go to 1.

Exponents

a^m = aaa…a m times	a raised to the m^{th} power is a multiplied times itself m times
$a^m a^n = a^{(m+n)}$	Multiplication with same base
$a^m \times b^m = (a+b)^m$	Multiplication with different base but same exponent
$(a^m)^n = a^{mn}$	Nested exponentiation
$\frac{a^m}{a^n} = a^{(m-n)}$	Division with **same bases**, different exponents
$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	Division with **different bases**, same exponents
$a^0 = 1$, $0^a = 0$, 0^0=?	Any number to the zero exponent is one, except 0^0 is undefined.
$a^{(-1)} = 1/a$	Any number to a negative one is the reciprocal of the number
$a^{½} = \sqrt{a}$	Square root of a number
$a^{1/n} = \sqrt[n]{a}$	The n^{th} root of a number a
$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$	$= \sqrt[n]{a^m}$, the n^{th} root of a raised to the m^{th} power.
if $a^m = a^n$, then m=n	If two sides of an equation have the **same base**, then there exponents must be equal.
$y = ca^x$, a>1	**Exponential growth** model since y increases exponentially.
$y = ca^x$, a<1	**Exponential decay** model since y decreases exponentially.
$y = ce^{bx}$, e=2.7183	Another form of exponential growth
$y = ce^{-bx}$, e=2.7183	Another form of exponential decay
$Pn = P_0(1+r)^n$, r = b-d	**Population growth model**, where P_0 is the population at time 0 and r is the net birth rate (birth rate – death rate), and P_n is the population size after n years. Note that this is exactly the same as the interest rate formula for compound interest with a different interpretation.

Set Notation

Set notation is one of those concepts that is very simple, but is made to seem much more complex when it is expressed in mathematical notation. Just remember that the notation is for compactness and precision. Try to get past the barrage of symbols and notation. A set is just a group of things having common attributes, e.g. all brown haired men.

A, B, C, …	Capital names used as names of sets
{a, b, c, …}	List of items in a set. Small letters are used for elements of sets
{ x | x = …}	The set of elements x such that x is equal to or has a given property
$a \in B$	a is an element of B
$a \notin B$	a is not an element of B
$A \not\subset B$	A is not a subset of B
$A \subseteq B$	A is subset of B
$A \subset B$	A is a proper subset of B, that is, B has a least one element which is not in A
A = B	The sets A and B have exactly that same elements
$A \cap B$	Intersection of sets A and B, sometimes written as a dot like multiplication
$A \cup B$	Union of sets A and B, sometimes written as a + like addition
A'or $\overline{A}$	The complement of set A, all elements not in A
N(A)=$\lvert A \rvert$	The number of elements of A.

De Morgan's Laws – for any two sets A and B

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

Percentages and Percent Change

$\left(\frac{p}{100}\right)x$	What is p% of x?
$\frac{x}{y}\times 100$	What percent is x of y?
$x=\frac{py}{100},\ y=\frac{100x}{p}$	If x is p% of y, what is the value of y? In later formulas the division by 100 will be understood.
$\frac{new-old}{old}\times 100$	What is the **percent change** from an old value to a new value?
$\frac{Selling\ Price-Cost}{Selling\ Price}\times 100$	What is the percent **markup**?
$\frac{Selling\ Price-Sale\ Price}{Selling\ Price}\times 100$	What is the percent **markdown**?
$x + xp = x(1+p)$	New value of x after **p % increase**
$x - xp = x(1-p)$	New value of x after a **p % decrease**
$x=\frac{y}{1+p},\ y=(1+p)x$	If y is the value of x after **p% increase** in x, what is the original value of x?
$x=\frac{y}{1-p},\ y=(1-p)x$	If y is the value of x after **p% decrease in x**, what is the original value of x?
$x(1+p)(1+q)$	**Multiple increases**. New value of x after p % increase and another q % increase
$x(1+p)^n$	New value of x after **n increases** of p %
$x(1+1) = 2x$	A 100% increase **doubles** of original value
$\frac{x}{(1-.5)x}=2$	A 50% decrease halves the original value and requires a 100% increase to get back to original value.
$px + (1-p)x = x$	Many percent problems involve the situation where p% of a total have a property or attribute and (1-p)% don't have that property, but together they add up to the total. It is convenient to call these "**p not p**" problems.
$\bar{p}=\frac{\sum_i^n p_i x_i}{\sum_i^n x_i}$	This is the **average percent** or the weighted average percent problem.

Algebra

<table>
<tr><td>$(+)\times(+)=(+)$
$(-)\times(-)=(+)$
$(-)\times(+)=(-)$
$(+)\times(-)=(-)$
$\frac{(+)}{(+)}=(+)$, $\frac{(-)}{(-)}=(+)$
$\frac{(+)}{(-)}=(-)$, $\frac{(-)}{(+)}=(-)$</td><td>Remember how the algebraic signs combine for multiplication and division.
a × b = (+) if and only if a and b both positive or both negative.
a × b = (-) if and only if a and b have opposite signs.

Similarly for division</td></tr>
<tr><td>-(1-x) = x-1
-(x-1) = 1-x</td><td>The negation of parenthetical expression reverses the signs of the enclosed terms.</td></tr>
<tr><td>$x^2 = a$</td><td>Remember that x can have two values: +√a and –√a.</td></tr>
<tr><td>Binomial</td><td>An expression that combines two terms: (a+b), (2x+3), etc.</td></tr>
<tr><td>Polynomial</td><td>An expression consisting of multiple terms: x^2+3x-4</td></tr>
<tr><td>Degree</td><td>The number of the highest exponent is the degree of the polynomial. For example, x^2+3x-4 is degree 2. The degree always specifies the number of values that can satisfy the equation of that degree.</td></tr>
<tr><td>$ax^2 + bx + c$</td><td>Any equation which has an x squared term x^2 .
Note that $ax^2 + c$ is also a quadratic equation.</td></tr>
<tr><td>a > 0</td><td>The graph is concave up.</td></tr>
<tr><td>a < 0</td><td>The graph is concave down.</td></tr>
<tr><td>(x+a)(x+b)
$x^2 + (a+b)x + ab$</td><td><table><tr><td></td><td>(a+b) +</td><td>(a+b) -</td></tr><tr><td>ab +</td><td>a+ b+</td><td>a- b-</td></tr><tr><td>ab -</td><td>a+ b-, $|a| > |b|$</td><td>a+ b-, $|a| < |b|$</td></tr></table></td></tr>
<tr><td>$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$</td><td>General solution to a quadratic equation. Gives two values for y = 0, where $y = ax^2 + bx + c$.</td></tr>
<tr><td>a(b+c) = ab + ac
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2-b^2$</td><td></td></tr>
<tr><td>$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$</td><td>The quantities (n k) are called the binomial coefficients and are defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. They also correspond to a count of the number of combinations of n things chosen k at a time. See permutations and combinations.</td></tr>
<tr><td>1
1 1
1 2 1
1 3 3 1
1 4 6 4 1</td><td>Pascal's Triangle is a pyramid shaped arrangement of numbers corresponding to the binomial coefficients. The numbers in each row are formed by starting with 1 and then adding the two numbers directly above it. The numbers in row n correspond to the binomial coefficients $(a+b)^n$.</td></tr>
<tr><td>-x = x → x =0</td><td></td></tr>
</table>

$ax + b = c$ $x = (c-b)/a$	Solving one equation in **one unknown** requires that x be isolated on one side of the equation.
$y + ax = b$ $x = c$ $y = b - ac$	Solving two equations with two unknowns by **substitution**.
$ay + bx = c$ $dy + bx = e$ $y = (c-e)/(a-d)$	Solving two equations with two unknowns by **elimination**.

Inequalities

$\neq < \leq > \geq$ $(x - a) < b \rightarrow$ $(a - x) > b$	**Inequalities.** Equations with inequalities can be treated like equalities except, when multiplying by a negative number, the direction of the inequality changes.
$\lvert x\rvert$	**Absolute value** returns a positive value if x is negative.
$\lvert x\rvert < a$	$\lvert x\rvert < a$ implies $-a < x < a \rightarrow x \in (-a, a)$
$\lvert x-a\rvert < b$	$\lvert x-a\rvert < b$ implies $-b < x-a < b$, $-b+a < x < b+a$
$x = -\lvert x\rvert$	x is negative or zero
$x/y < 1$ $s(x) = s(y), xy > 0$ $s(x) \neq s(y), xy < 0$	1) When x and y have opposite sign ($xy < 0$) 2) When x and $y > 0$ (pos) and $x < y$ 3) When x and $y < 0$ (neg) and $\lvert x\rvert < \lvert y\rvert$, or $x > y$ 4) Note that $-4 < -2$, but $-4/-2 > 1$
$x^2-1 > 0$	$x < -1$ and $x > 1$, $[-\infty, -1)$ $(1, \infty]$
$1-x^2 \geq 0$	$[-1, 1]$

Equation for a straight line and linear relationships

1) y = mx + b 2) y = m(x-a)	**y-intercept form (1)**: m = slope and b = y intercept for the point (0,b) **x-intercept form (2)**: m = slope and a = x intercept for the point (a, 0) Any linear equation can always be put into the slope intercept form. Slope m is equal to the rise divided by the run = the change in y divided by the change in x = Δy/Δx.
$m = \frac{y_1 - y_2}{x_1 - x_2}$	**When two points are given, the slope of the line is given as shown.**
$y - y_1 = m(x - x_1)$	**Point-slope form** of straight line, where m = slope and (x_1,y_1) is any point on the line. This relationship follows from the fact that the slope m is equal to $(y-y_1)/(x-x_1)$.
$y_1 - mx_1$	**Y-intercept – value for b in y-intercept form**
$x_1 - \frac{y_1}{m}$	**X-intercept – value for a in x – intercept form**
$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$	**Two-point form** of straight line, where (x_1,y_1) and (x_2,y_2) are any two points on the line. This relationship follows from the fact that the left and right side of the equation both represent slopes from the same line.
$m_1 = m_2$	Two lines with slopes m_1 and m_2 are **parallel** if their slopes are equal.
$m_1 m_2 = -1$	Two lines with slopes m_1 and m_2 are **perpendicular** if their slopes are the negative reciprocal on each other.
y = kx	y varies directly with x, or y is proportional to x. k is the constant of proportionality.
y = k/x	y varies inversely with x, or y is inversely proportional to x

Statistics – Measures of Central Tendency

$\bar{x} = \dfrac{\sum_i^n x_i}{n} = \dfrac{1}{n}\sum_i^n x_i$ $n\bar{x} = \sum x_i$	**Average value = mean = expected value** The average value of x is the sum of n x values divided by the number of values n. You can think of 1/n as the uniform probability that x will occur if it is a randomly selected value. The mean and other measures of central tendency are used to summarize data. To determine how well the mean describes the data, we must look at the variability of data around the mean. Note that the **sum = (average) × (number of terms)**
$\bar{x}$	Mean of **sample** is written as x with a bar over it.
$\sum_{i=1}^n (x_i - \bar{x}) = 0$	The sum of the differences about the mean always total to zero. This is a fundamental property of the mean.
μ	Mean of **population** is written as the Greek letter mu.
Mode	The value of x which occurs most frequently is called the **mode**. It is by definition the most common (or typical) value observed.
Median x_i must be ordered $x_{\frac{n+1}{2}}$, n odd $\dfrac{x_{\frac{n}{2}} + x_{\frac{n+2}{2}}}{2}$, n even	The value of x which has half of the value above it and half of the values below it is called the **median**. If x_i is an ordered list of all values of x, and n is odd, then $x_{(n+1)/2}$ is the median. The median is the same as the 50^{th} percentile. If there is an even number of data points then the median is the average of the two middle values of x. The median has the desirable property that it is not affected by extreme values as is the mean.
mean = mode	Distribution is **symmetric**
median < mean	Distribution is **skewed to the right** (mean-mode > 0)
mean < median	Distribution is **skewed to the left** (mean-mode < 0)
$\bar{x} = \dfrac{\sum_i^n w_i x_i}{\sum_i^n w_i}$	**Weighted Mean** The common example is grade point average (GPA). Another example is computing the average value of x when the x's are grouped by frequency. See the following entry.
$\bar{x} = \dfrac{\sum_i^n x_i f(x_i)}{\sum_i^n f(x_i)}$	Note that $\dfrac{f(x_i)}{\sum f(x_i)} = p(x_i)$. The subscript in this formula can be easily confused with the subscript for all x's. This formulas only indexes the unique values of x, while the original formula indexes all values of x. Thus the value for n in this case is the number of unique x's, not the total number of x's.
$E(x) = \sum_i^n p_i x_i$	The **expected value** of x is the sum of the x's weighted by their probabilities. Note that dividing by the sum of the p's is the same as dividing by 1 since probabilities must sum to one. The average value formula above is the same as the expected value formula with pi = 1/n. In other words, each of the x's is assumed to occur with equal probability.

Statistics – Measures of Variability

Range = max – min	Difference between largest and smallest value
$x_{nQ} = Q^{th}\ Percentile$	**Percentiles and quartiles** – If x_i are sorted from lowest to highest, the element x_i is the Q^{th} percentile if i = floor(nQ).
$\lvert x \rvert$	The **absolute value** of x is written as x with two horizontal bars on each side. The absolute value always returns a positive value if x is negative, e.g. |-3| = +3.
$\bar{d} = \frac{1}{n}\sum_{i}^{n} \lvert x_i - \bar{x} \rvert$	The **mean deviation** is the sum of differences of each x value form the mean value. It measures the average amount by which a set of numbers differ from the mean. It is the first measure of the confidence we can have that our mean or average is a good descriptor of the data.
$v = \frac{1}{n}\sum_{i}^{n}\left(x_i - \bar{x}\right)^2$	The variance is the sum of mean differences squared. This is similar to the mean deviation except we have gotten rid of the clumsy absolute value operators by squaring the differences. Of course this value is now much larger than we would like so will take the square root of it to create the standard deviation which is much closer to the md.
$v = \frac{\sum x_i^2}{n} - \bar{x}^2$ $v = \frac{1}{n}\sum x_i^2 - \bar{x}^2$ $v = \frac{1}{n^2}\left(n\sum x_i^2 - \left(\sum x_i\right)^2\right)$	Computational form of variance. When the sample variance is defined with a denominator of n rather than (n-1), the estimator is biased, and one gets the result that $E(S^2) = \frac{n-1}{n}\sigma^2$
S = √v	The **standard deviation** is the square root of the variance. It gets the measure of variability closer to the mean deviation, but it will always be bigger than the mean deviation, but with much nicer mathematical properties.
σ	The standard deviation of the population is written as the Greek letter sigma to distinguish it from the sample standard deviation s.
$y - \bar{y} = m(x - \bar{x})$ $m = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum x^2 - \left(\sum x\right)^2}$	The **regression line** is the straight line which minimizes the square of the distances from the points to the line. In the adjacent equation, m is the slope of the line. Note the point $(\bar{x}, \bar{y})$ is on the line.
$-1 < r < 1$	**Correlation coefficient** "r" measures how well a straight line describes the relationship between the variables. The closer |r| is to one, the better the fit. $r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{\left(n\sum x^2 - \left(\sum x\right)^2\right)\left(n\sum y^2 - \left(\sum y\right)^2\right)}}$
$0 \le r^2 \le 1$	**Coefficient of determination** = the fraction of total variation explained by the least-square regression line.

Probability

P(A) $P(A)=\frac{\lvert E\rvert}{\lvert S\rvert}$	The **probability of A** occurring is P(A). Equivalent terminology is "**chance of A**" or "**likelihood of A**" Using the vertical lines to indicate the number of elements S = {sample space set of all possible outcomes} E = {event set which is a subset of S} An experiment is a action to generate an event.
$0 \le p_i \le 1$ $\sum_{i=1}^{n} p_i = 1$ $P(\varnothing)=0$	The set of numbers p_i is a **probability distribution** if it satisfies the following properties: 1. All probabilities p_i must be in the interval (0,1) 2. The sum of all the probabilities p_i must exactly one. If there are n probabilities, then $p_1 + p_2 + p_3 + \ldots + p_n = 1$ 3. Probability of the empty set is 0.
P(A and B) = P(A)P(B)	**Independent Events** If A and B are independent events, then the probability of both A and B happening is equal to the product of the probabilities.
$P(A\cup B) = P(A) + P(B)$	If two sets of events are **mutually exclusive or disjoint (**no elements in common) then the probability of either A or B happening is the sum of the probabilities. $P(A\cap B) = 0$ for mutually exclusive events
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$	**Non Mutually Exclusive Event Sets** This formula works because if A and B are mutually exclusive then $P(A\cap B)$ equals 0.
$P(A \mid B)=\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$ $P(A/B)=\frac{\lvert A\cap B\rvert/\lvert S\rvert}{\lvert B\rvert/\lvert S\rvert}$ $P(A\mid B) = P(A\cap B) / P(B)$	**Conditional Probability of A given B** Notice that you can view conditional probability of A given B as if B was the new sample space. If you divide both the numerator and the denominator by the outcomes in |S|, you arrive at the probabilistic version of the formula. They are different versions of the same thing.
P(a) = 1/n P(a and b) = P(a)P(b) $= 1/n^2$	A is **uniform random number** if the probability of every possible a is equally likely and the occurrence of any pair of numbers a and b is independent. E(X) = (n+1)/2.
$a:b=\lvert A\rvert:\lvert\overline{A}\rvert$ $P(A)=\frac{\lvert A\rvert}{\lvert A\rvert+\lvert\overline{A}\rvert}=\frac{\lvert A\rvert}{\lvert S\rvert}$	**Odds** of A = outcomes in A : outcomes is not A Let a = |A| and b = |not A|. These counts are just like frequencies of the events happening which can be converted to probabilities by dividing by the total of the frequencies.
$P(E)=\frac{a}{a+b}=p$ $P(E')=\frac{b}{a+b}=1-p$	If the **odds in favor** of an event are a:b, then the probability of the event E is a/(a+b). The probability of against E are b:a. For example, if odds in favor of E are 3:2, the p=3/5 and 1-p=2/5.
a:b = na:nb	The odds are not changed if they are multiplied by a constant.

$p:1-p$ if 1-p=q p:q	If event E has probability p, then not E has probability 1-p and the odd in favor of E can be written as p:1-p. These numbers can usually be converted to integers by multiplying by an appropriate constant. For example 1/4:3/4 = 1:3

Probability Distributions - Discrete

$p_i = 1/n$	This is the **uniform distribution** of n events where each event has an equal change of occurring (1/n). For example **flipping a coin** has two events (heads and tails) with equal chances (probabilities) of occurring ($p_i = 1/2$). Note that when a coin is flipped a repeated number of times, the number of heads occurring in n flips has a binomial distribution. The **six faces of a die** has a uniform distribution with $p_i = 1/6$. Each face has the same chance of occurring. **Random numbers** from 0 to 9 have a uniform distribution with $p_i = 1/10$.
$\binom{n}{x} p^x q^{(n-x)}$	**Binomial distribution** = P(x successes) where q = 1-p. Note that $\mu = np$ and $\sigma^2 = npq$. For large values of n, the binomial distribution can be approximated by the normal distribution.

Probability Distributions – Continuous

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$	Normal distribution with mean μ and standard deviation σ.

Counting, Permutations, and Combinations

$r_1 \times r_2 \times r_3 \times \ldots \times r_k$	**Fundamental Rule of Counting – Pizza Principle** For a sequence of events $A_1, A_2, A_3, \ldots, A_k$, if the event A_1 can occur r_1 ways, the event A_2 can occur in r_2 ways for each of the r_1 ways of A_1, and so on, then the number of ways that A_1 through A_k can occur, in succession, is the product of the number of elements in each set A_i.
n^k	The number of ways to **arrange k objects from a set of n distinct objects** when you can use an object more that once. This is called **ordered sampling with replacement.**
n^k	The number of ways to **arrange n outcomes in k independent experiments** such as flipping a coin or rolling a die, e.g. three outcomes of flipping a coin generate 2^3, and three outcomes of rolling a die generate 6^3 arrangements or outcomes.
$n! = n(n-1)(n-2) \ldots 1$ $0! = 1$	**n factorial** is the product of n times each integer less than n times itself. n! is equal to the number of ways to arrange n distinct things using each object exactly once. If you only use a subset of the objects, the result is called a permutation as described below $_nP_n$
$n! = n(n-1)!$	**Recursive definition** of n factorial
$n(n-1) = n! / (n-2)!$	**A factorial identity**
$0! = 1$	An important definition to know.
"arrange n" $n!$	The number of ways to **arrange n objects** is given by n!. That is, you can arrange the first position n-ways, the second position (n-1) ways, the third position (n-2) ways and so on. Note that this is a special case of $_nP_k$ below where k = n.
$\dfrac{n!}{k_1!k_2!k_3!\cdots}$	The number of ways to arrange n objects when some of the objects are the same; namely, k_1 are the same, k_2 are the same, k_3 are the same, etc. For example, the number ways to arrange AAABCDD is given by 7!/(3!2!). Or, the number of ways that values 1, 1, 1, 6 can be rolled by four dice is 4!/3!. This can also be interpreted as the number of ways to partition a set of n objects into m cells, where m is the subscript on k. For example, the number of ways that seven students can be assigned to one triple and two double hotel rooms is given by 7!/(3!2!2!).
"n arrange k" $_nP_k = \dfrac{n!}{(n-k)!}$ $_nP_n = n!$	The number of number of n different things taken k at a time is called $_nP_k$ **permutations** of n. This can also be stated as the number of ways to arrange k objects from a set of n distinct objects using each object at most once. This is called **ordered sampling without replacement**. This is sometime spoken as "n arrange k." Note that $_nP_k = n!$ if k=n, i.e. you use the entire set of objects rather than a subset of the objects. Remember $(n-n)! = 0! = 1$.

"n choose k" $_nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ Number of subsets of size k from n objects	The number of ways of choosing k objects from a set of n distinct objects is called the **combinations** of n things taken k at a time and is spoken as: "n choose k." Note that if you choose the entire set (k=n) then you get the entire set and the number of combinations is exactly 1. This can be thought of as the number of subsets of size k that can be formed from n elements. This is called **unordered sampling without replacement**. This is also the binomial coefficient as described above.
$\binom{n}{0} = \binom{n}{n} = 1$	Definitions which rely on 0! = 1.
$\binom{n}{k} = \binom{n}{n-k}$	Simple identity.
$\sum_{k=0}^{n} \binom{n}{k} = 2^n$	The sum of the binomial coefficients, which is also the number of subsets of n items.
$_nC_k = {}_nP_k / k!$	Relationship between combination formula and permutation formula.
2^n	The number of subsets in a group of n distinct objects.

Summary of Counting Formulas

	Ordered	**Unordered**
With Replacement	**Arrangements of k items** $\mathbf{n^k}$	Not meaningful
Without Replacement	**Permutations** "n arrange k" Whole Set Subset **n!** **n! / (n-k)!** Some objects the same **n!/k!**	**Combinations** "n choose k" Subset only **n! / k! (n-k)!**

Summation Formulas

$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$	Basic summation notation. This is just a short hand way of writing a long complicated process. It is an example of what mathematicians do best, namely finding compact, sometimes abstract notations to described processes that are frequently encountered.
$\sum_{i=1}^{n} a = na$	Adding up a constant n times.
$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$	Adding up the first n integers. Consecutive integers beginning with n, can always be written as n, n+1, n+2, … The number of consecutive integers from a to b inclusive is a+b+1.
$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$	The sum of the first n integers is equal to this amazing formula discovered by Gauss. Equals the average of the first n integers (n+1)/2 multiplied by the number of integers n.
$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i$	In this case, it doesn't matter if you start at 0 or 1.
$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$	If you only sum up to (n-1), it is as if n = n-1.
$\sum_{i=1}^{2n} 2i - 1 = n^2$	Sum of first n odd integers
$\sum_{i=1}^{2n} 2i = (n+1)n$	Sum of first n even integers
$\sum_{i=n}^{k-1} i =$ $n + (n+1) + (n+2) + \cdots + (n+k-1)$ $= kn + \sum_{i=0}^{k-1} i = kn + \frac{(k-1)k}{2}$	Sum of k integers beginning at n. This is just a variation of the previous formulas.
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	The sum of the first n integers squared.
$\sum_{i=1}^{n} i^3 = \frac{(n+1)^2 n^2}{4}$	The sum of the first n integers cubed.
a, a+x, a+2x, a+3x … $S = \sum_{i=0}^{n-1} (a + xi) = \frac{n}{2}(a + L)$ $S = \frac{n}{2}(2a + (n-1)x)$	Arithmetic Progression and summation. Each term increases by a constant amount x. In the adjacent equation, L is the value of last term. You can think of the sum as the average of all the numbers (a+L)/2 multiplied by the number of numbers n. Since $L = a + (n-1)x$, the alternative sum formula can be derived.

a, ax, ax^2, ax^3 … $\sum_{i=0}^{n-1} ax^i = a\frac{1-x^n}{1-x}$	Geometric Progression and summation
$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \lvert x \rvert < 0$	Infinite series summation
$\sum_{i=0}^{n-1} ax^i = a\frac{x^n-1}{x-1}$	Sum of geometric series

Interest Formulas

PV, FV, n, r (No Payments)	
$FV = PV(1+nr)$	Future value FV of a present value PV at **simple interest** r for n years (periods).
$FV = PV(1+r)^n$	Future value F of a present value P at **compound interest** r for n years (periods).
$FV = PV(1+r/k)^{kn}$	Future value F of a present value P at **compound interest** r for n years (periods) when the compounding takes place k times per year. For example k=4, and k=12 are common.
$FV = PV\ e^{nr}$	Future value F of a present value P with **continuous compounding** at interest rate r for n years.
IAF	The interest rate adjustment factor (IAF) is the factor that converts between PV and FV, $(1+r)^n$.
EAR	Effective annual rate (EAR) = IAF – 1. In this case, IAF is usually equal to $(1+r/n)^{12}$, where n is the compounding periods per year.
$e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$	The definition of the number e.
$r = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1$	The **interest rate** r that will grow an amount PV to an amount FV in n periods.
$n = \frac{\log\left(\frac{FV}{PV}\right)}{\log(1+r)}$	The **number of periods** n needed to grow an amount PV to an amount FV at an interest rate r.
$n = \frac{\log 2}{\log(1+r)}$	The number of periods needed to **double money** at an interest rate r.
PV, FV, s, n, r (Savings Formulas)	
$FV = s\frac{(1+r)^n - 1}{r}$ $s = \frac{rFV}{(1+r)^n - 1}$	**Savings Future Value (Normal) .** Future value of an ordinary "savings" annuity with payment "s" at the end of the period with interest rate r for n periods. r is normally equal to "annual rate ÷12", and n is normally equal to "years × 12."
$FV = s(1+r)\frac{(1+r)^n - 1}{r}$	**Savings Future Value (Annuity Due)**. Future value of an **annuity due** with payment "s" at the beginning of the period with interest rate r for n periods. Notice that an annuity is just shifted by one period or by $(1+r)^n$. Not commonly used.
PV, FV, m, n, r (Loan Formulas)	
$m = \frac{PV(1+rn)}{n/12}$ n = months	**Loan Payment for add-on interest r**. Monthly payment m for a loan of an amount PV with add-on interest. Note that loans work with PV's and savings work with FV's. The future value of a loan is zero. Remember that r is the annual rate and n is the number of months of the loan (years × 12).

$m = \frac{PVr(1+r)^n}{(1+r)^n - 1}$ $PV = \frac{m((1+r)^n - 1)}{r(1+r)^n}$	**Loan Payment for amortized loan.** Monthly payment m for a loan of amount PV which **amortizes** at interest rate r. This is the rate associated with **APR** where interest is charged only on the **unpaid balance** of the loan. Note that there are no simple formulas for r and n. These values must be found with tables or iteratively on a calculator or computer. This also the formula for an **income annuity**.
$m = \frac{PVr}{1 - \frac{1}{(1+r)^n}}$	**Loan Payment – Calculator Format.** This formula is the same as the one above but expressed in a different format that facilitates the computations on a simple calculator without writing down intermediate values or using the store function. Notice that in this form, it is easy to see that the as the number of payment periods n becomes very large, the monthly payment converge to an interest only loan with a single balloon payment sometime in the future.
$m = s(1+r)^n$	**Savings to loan conversion factor**. The monthly payment for a loan of x dollars is equal to the monthly savings amount to reach x dollars multiplied by the IAF.
$PV(1+r)^i - \frac{m\left[(1+r)^i - 1\right]}{r}$	The **unpaid balance** of a loan at the end of period i.
$ABD = \frac{\sum_{i=1}^{n} B_i D_i}{\sum_{i=1}^{n} D_i}$	**Average Daily Balance** for credit card calculations, where B_i is the balance that exists for D_i days.
APR	**Annual Percentage Rate.** APR is the interest rate that would generate an equivalent monthly loan payment if interest were charged only on the unpaid balance during loan payback. To find APR, you must know the loan and the payment amounts. You must then iteratively test alternative rates until you find a rate which will give a monthly payment acceptably close to the stated payment.
PITI	**Monthly home mortgage payment** base upon the sum of **principal, interest, taxes, and insurance**. Principal and interest (PI) are typically computed as a single number equal to the monthly loan payment for a normal amortized loan at the specified rate and term. Taxes and insurance must be converted to monthly amounts and added to the loan payment to get the monthly amount sent to the mortgage company, commonly called the PITI amount.
PV, FV, a, n, r (Income Annuity)	
$a = \frac{PVr(1+r)^n}{(1+r)^n - 1}$	**Income Annuity.** This is the amount of money a that you could withdraw for n periods if you had PV invested at interest rate r. Note that this is the same formula used for loans.

Conventions Concerning Letters and Symbols

r, s, t, u, v, w, x, y, z	Small letters at the end of the alphabet are commonly used to represent unknown variables. "r" is commonly used for interest rate since "i" is easily confused with a subscript.
a, b, c, d	Small letters at the front of the alphabet are commonly used to represent unknown constants.
i, j, k, m, n	Small letters in the middle of the alphabet are commonly used to represent indexes of variables in the form x_i and y_i. The small letter "L" is usually not used because it confused with the number "1."
A, B, C, D, E	Capital letters at the beginning of the alphabet are common used to represent sets.
π = circumference λ = wave length μ = mean of population σ = sd of population	Small Greek letters are commonly used to represent mathematical or physical constants, angles, and special variables.
$\in$ element of $\notin$ not an element of ∞ infinity = equal $\equiv$ identity $\neq$ not equal $\approx$ approximately equal $\supset$ subset of $\subset$ subset of $\sum$ summation $\prod$ product $\angle$ angle $\cap$ intersection $\cup$ union $\pm$ plus or minus $\surd$ square root $\div$ division	Capital Greek letters are commonly used to represent mathematical operations. You can think of most of these as "mathematical verbs." $\therefore$ therefore $\ni$ such that $\exists$ there exists Δ change or difference $\int$ integration $\|\ \|$ absolute value $\|\ \|$ number of $\|$ set builder (such that) { } elements of a set ( ) open interval [ ] closed interval